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Modified solenoid scattering for the Aharonov–Bohm effect

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Abstract. A method of partial-wave analysis is used to extend the treatment of Aharonov and Bohm for the scattering from an unscreened solenoid of infinitesimal radius to include screening and non-zero radius. The method yields a scattering amplitude which bears a formal resemblance to that appropriate to modified Coulomb scattering. The scattering amplitude and the momentum transfer cross section retain a dependence on the enclosed flux as the screening barrier becomes infinite, resulting in a component of force, periodic in the enclosed flux, being exerted on the barrier. The resulting implications concerning the locality of electromagnetic interactions are discussed. The force is shown to persist when the conditions of strict impenetrability are relaxed and the possibility of directly observing this quantum force is examined.

1. Introduction

The effects (henceforth AB effects) of 'enclosed' electromagnetic fluxes, on charges which are excluded by barriers from entering the flux-containing regions, first achieved prominence in a paper by Aharonov and Bohm (1959) (henceforth referred to as AB). Most of the debate (see Olariu and Popescu (1985) (henceforth referred to as OP) for a comprehensive review) stemming from this provocative paper, and all the associated experimental work (OP, § III) has been concerned with interference effects. Although a non-zero momentum‡ transfer cross section, periodic in the enclosed flux, was a striking implication of the AB analysis of the solenoid scattering problem which they used to illustrate their arguments, no attempt to detect such forces experimentally, nor to theoretically assess their detectability, has been reported. To some extent this reflects the limitations of the AB 'magnetic string' model which is valid (Brown 1985) only for electron wavelengths which are very long compared to the solenoid radius, a condition which appears experimentally unapproachable. In addition, the absence of a shielding barrier in this model meant that a local mechanism to account for the force was lacking, leaving the reality of its existence open to doubt.

More realistic models of a solenoid surrounded by cylindrical barriers of non-vanishing dimensions have been considered by Kretzschmar (1965), Berry *et al* (1980), Olariu and Popescu (1983), Brown (1985) and OP (§ II F), but the form of the scattering amplitude, momentum transfer cross section, etc, was not investigated. Peshkin *et al* (1961) and Tassie (1963) have considered two-cylinder and toroidal configurations,

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‡ Throughout this paper 'momentum' means kinetic momentum, i.e. the product of mass and velocity.

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respectively, using symmetry arguments to study the case of half-integral flux parameter, α; they showed that the total cross sections (which are finite for the configurations studied) displayed a periodic dependence on the enclosed flux. Despite such demonstrations the situation remains clouded—Olariu and Popescu (1983) have recently claimed that there is no momentum transfer for the magnetic string case but (OP, p 372) have more recently revised this opinion. The assessment by OP of the matter is somewhat confusing. In § I G of their review they conclude that the 'total kinetic momentum is not changed by distributions of enclosed electromagnetic fields'. Such a conclusion, based on their equation (1.89), is valid only if *all* electromagnetic fields are enclosed—a situation which is physically empty since the fields associated with excluding barriers themselves, and/or slits, must be accessible to the electron undergoing a scattering or interference process in order to be effective, whether or not there is any enclosed flux. In general, enclosed flux can influence momentum transfer through the mediation of other (accessible) fields $\boldsymbol{E}$ or $\boldsymbol{B}$. This mechanism, which is not new (see Peshkin *et al* 1961, Aharonov and Bohm 1961) is strikingly illustrated for the scattering problem considered in the present paper by a version of Ehrenfest's theorem (Brown 1986) which relates the momentum transfer cross section σ_{tr} directly to the expectation value (with respect to the scattering wavefunction ψ) of the Lorentz force. Expressed by (5.2) this shows (in the solenoid case) that even if $\psi = 0$ wherever $B \neq 0$ the magnetic flux still influences σ_{tr} via the field $\boldsymbol{E}$ associated with the excluding barrier, because ψ depends (through the Schrödinger equation) on the vector potential field which is everywhere non-vanishing. It is presumably such effects to which OP refer in the final paragraph of their § II F.

The aim of the present paper is to provide a general solution of the screened-solenoid problem, permitting it to be discussed in a wavelength range which is experimentally feasible and allowing the quantum effects of the flux to be gauged when the condition of impenetrability of the barriers is relaxed. Our analysis is based on the partial wave treatment of Kretzschmar (1965) but also preserves strong links with the original solution of AB.

The scattering solution for a rectangular screening potential barrier is obtained in § 2. Its asymptotic form is identical to that of the AB magnetic string solution but the scattering amplitude depends on the details of the screening in a way which bears a formal resemblance to the case of 'modified Coulomb scattering' (Joachain 1975, § 6.3). The extension of this analysis to apply to any screening potential of finite range is briefly sketched. As the screening potential becomes infinite (§ 3) the scattering amplitude displays an AB effect, i.e. it retains a dependence on the enclosed magnetic flux and this dependence becomes periodic, with the period h/e of the flux quantum. The same is true of σ_{tr}, resulting in the shielding barrier experiencing an additional force depending periodically on the enclosed flux. In § 4 this force is further discussed with regard to its continued existence when the condition of strict impenetrability of the barrier is relaxed, and with regard to its possible observation. The role of the local Maxwellian fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in mediating this flux-dependent quantum force is investigated in § 5 and our conclusions are briefly summarised in § 6.

2. Partial wave analysis of solenoid scattering

The aim of this section is to extend the analysis of AB for scattering from an unscreened magnetic string to the case of a screened solenoid of non-zero radius r_0. To obtain mathematical tractability and to preserve the framework used by most other workers we retain the idealisation of infinite solenoid length L. This may be justified by assuming that all the apparatus (filament, electron gun, detectors, etc) involved in the conceptual scattering experiment lies within some finite region $\mathscr{S}$ filling the space $r \leq R$ between the axial limits $-b \leq z \leq b$. By taking finite $L \gg \max(R, 2b)$ we can reduce the magnetic induction field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ throughout $S_0(R \geq r \geq r_0;\ -b \leq z \leq b)$ to arbitrarily small values and at the same time ensure that further increases in L have arbitrarily small effects on the values of $\boldsymbol{B}$ and $\boldsymbol{A}$ throughout $\mathscr{S}$. We may conclude that letting $L \to \infty$ cannot materially affect the motion of electrons in $\mathscr{S}$; to conclude otherwise would imply a type of non-locality which is far more extreme than has previously entered discussions of the AB effect and which evidently would violate causality.

Following Kretzschmar (1965) and Brown (1985) we consider the scattering solutions $\psi(r, \theta)$ for the energy eigenstates of the two-dimensional system described by

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial}{\partial \theta}-\frac{irq}{\hbar}A_\theta\right)^2+k^2-\frac{2mV}{\hbar^2}\right]\psi=0. \tag{2.1}$$

This equation applies to a charge q having mass m and energy $E = \hbar^2k^2/2m$ moving under the combined influences of an electrostatic potential (energy) field V and a magnetostatic vector potential field $\boldsymbol{A}$ with components (in Coulomb gauge and SI units)

$$\begin{aligned} A_r &= 0 \qquad & A_\theta &= \phi/2\pi r \qquad & r &\geq r_0 \\ & & &= \phi r/2\pi r_0^2 \qquad & r &\leq r_0. \end{aligned} \tag{2.2}$$

We henceforth assume† for simplicity and definiteness that

$$\begin{aligned} V \equiv v(\hbar^2k^2/2m) &= \text{constant} \qquad & r &\leq r_0 \\ &= 0 \qquad & r &> r_0 \end{aligned} \tag{2.3}$$

although, as discussed below, our results may be proved for any potential function $V(r)$ which is of 'finite range', i.e. for which V vanishes if r exceeds some finite value.

To solve (2.1) we write quite generally

$$\psi(r, \theta) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta}\, \xi_n(r) \tag{2.4}$$

and find in the region outside the solenoid

$$\xi_n(r) = a_n J_{|n+\alpha|}(kr) + b_n Y_{|n+\alpha|}(kr) \qquad r \geq r_0 \tag{2.5}$$

where $\alpha = -q\phi/h$ and $J(Y)$ denote Bessel functions of the first (second) kind. The continuity of ψ and $\nabla\psi$ at $r = r_0$ yields

$$-\frac{b_n}{a_n} = \frac{kA_nJ_{|n+\alpha|}(kr_0) - kJ'_{|n+\alpha|}(kr_0)}{kA_nY_{|n+\alpha|}(kr_0) - kY'_{|n+\alpha|}(kr_0)} \equiv R_n(kr_0, \alpha, v) \tag{2.6}$$

where $J'_\nu \equiv \partial J_\nu(x)/\partial x$, etc, and $kA_n = kA_n(kr_0, \alpha, v)$ is the logarithmic derivative of the 'interior' ($r \leq r_0$) radial partial wavefunction evaluated at $r = r_0$; it is given by (cf Brown 1985, equation (A1.1))

$$kr_0A_n = |n| - \alpha + 2\alpha\left(\frac{\partial}{\partial z}\ln \Phi(d_n, c_n, z)\right)_{z=\alpha} \tag{2.7a}$$

† Such a barrier may be approximately constructed by surrounding the solenoid by a coaxial cylindrical capacitor with plates at r_0 and $r_0 + \delta r$, $\delta r \ll r_0$.

where

$$2d_n = 1 + n + |n| - k^2 r_0^2 (1 - v)/2\alpha \qquad c_n = 1 + |n| \tag{2.7b}$$

and

$$\Phi(d, c; z) = 1 + \frac{d}{c} z + \frac{d(d+1)}{c(c+1)} \frac{z^2}{2!} + \ldots \tag{2.7c}$$

is the confluent hypergeometric series (Erdelyi *et al* 1953, § 6.1). The values of a_n and b_n are further restricted by requiring that (2.4) asymptotically represent, when multiplied by $\exp(-\mathrm{i}Et/\hbar)$, a superposition of a wave of spatially constant amplitude (representing the incident beam) and an outward propagating 'scattered' wave. We show in the remainder of this section that the choice

$$a_n = \cos \Delta_n \exp(\mathrm{i}\mu_n) \qquad b_n = -\sin \Delta_n \exp(\mathrm{i}\mu_n) \qquad \mu_n = \delta_n + n(\tfrac{1}{2}\pi - \theta_k) \tag{2.8}$$

in which θ_k is an arbitrary constant which represents the direction of propagation $\hat{\boldsymbol{k}}$ of the incident beam, satisfies this requirement. Here

$$\delta_n = \delta_n^\alpha = \delta_n^{\mathrm{AB}} + \Delta_n \tag{2.9a}$$

where

$$\delta_n^{\mathrm{AB}} = (n - |n + \alpha|)\pi/2 \tag{2.9b}$$

denotes the phase shift in the nth partial wave (relative to the cylindrical components of the plane wave $\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})$). The 'core' contribution $\Delta_n = \Delta_n^\alpha$ due to the non-vanishing kr_0 and v is defined by

$$\tan \Delta_n = R_n(kr_0, \alpha, v) \tag{2.10a}$$

and it is evident from the way Δ_n enters (2.8) that this modulo-π definition is sufficient. It follows from (2.7a) and Kummer's transformation (Erdelyi *et al* 1953, § 6.3) that $A_n(-\alpha) = A_{-n}(\alpha)$ and hence, using (2.6) and (2.10a), that

$$\Delta_n^{-\alpha} = \Delta_{-n}^{\alpha}. \tag{2.10b}$$

We introduce the AB magnetic string solution†

$$\psi^{\mathrm{AB}} = \psi_k^{\mathrm{AB}}(r, \theta; \alpha) = \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}n(\theta - \theta_k + \pi/2)] \exp(\mathrm{i}\delta_n^{\mathrm{AB}}) J_{|n+\alpha|}(kr) \tag{2.11}$$

and use it with (2.8), (2.5) and (2.4) to find, for $r \geqslant r_0$,

$$\psi = \psi_k(r, \theta; \alpha) = \psi^{\mathrm{AB}} + \mathrm{i} \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}n(\theta - \theta_k + \pi/2)] \exp(\mathrm{i}\delta_n) \sin \Delta_n H_{|n+\alpha|}^{(1)}(kr) \tag{2.12}$$

where $H^{(1)} = J + \mathrm{i}Y$ and the convergence of the sums in (2.11) and (2.12) may readily be confirmed.

The asymptotic behaviour of ψ for $kr \gg 1$ may be studied using (2.11) and (2.12). We first note the asymptotic form (Watson 1944, § 7.21)

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + \mathrm{i}Y_\nu(x) \sim (2/\pi x)^{1/2} \exp\{\mathrm{i}[x - (\nu + \tfrac{1}{2})\pi/2]\} + \mathrm{O}(x^{-3/2}) \tag{2.13}$$

† On using (2.9) and setting $\theta_k = \pi$, as considered by AB, (2.11) yields the solution which was studied by AB.

in which the remainder terms are negligible provided $|x| \gg |\nu|^2$; hence it can be usefully applied only to those terms of (2.11) and (2.12) for which $|n+\alpha|^2 \ll kr$. Choosing any integer $M \geq 1$ and satisfying

$$|M| \geq |\alpha| \qquad \text{and} \qquad |M|-|\alpha| \ll (kr)^{1/2} \tag{2.14}$$

we partition (2.12) according to

$$\sum_{n=-\infty}^{\infty} = \sum_{|n| \leq M-1} + \sum_{-\infty}^{-M} + \sum_{M}^{\infty}. \tag{2.15}$$

For each term of the finite sum in (2.15) we may use (2.13). Then (2.12) becomes

$$(\psi - \psi^{\text{AB}}) r^{1/2} \exp(-\mathrm{i}kr) \sim f_{M-1}(\theta-\theta_k) + \left(\sum_M^{-} + \sum_M^{+} \right) r^{1/2} \exp(-\mathrm{i}kr) + \mathrm{O}(r^{-1}) \tag{2.16}$$

where

$$f_{M-1}(\theta-\theta_k) = \frac{\exp(-\mathrm{i}\pi/4)}{(2\pi k)^{1/2}} \sum_{|n| \leq M-1} \exp(2\mathrm{i}\delta_n^{\text{AB}})[\exp(2\mathrm{i}\Delta_n)-1] \exp[\mathrm{i}n(\theta-\theta_k)] \tag{2.17}$$

and where

$$\sum_M^{\pm} = \mathrm{i} \sum_{\pm M}^{\pm\infty} \exp[\mathrm{i}n(\theta-\theta_k+\pi/4)] \exp(\mathrm{i}\delta_n) \sin \Delta_n H^{(1)}_{|n+\alpha|}(kr). \tag{2.18}$$

In appendix 1 we show that not only does $\Delta_n \to 0$ as $|n| \to \infty$, thereby ensuring the convergence of (2.17) as $M \to \infty$, but also that it vanishes sufficiently fast to ensure that $r^{1/2}(\Sigma_M^- + \Sigma_M^+) < \varepsilon$ for any $\varepsilon > 0$ provided $M > M'(\alpha, kr_0, v, \varepsilon)$ which is independent of r when kr exceeds some x_α. As a result we can proceed directly to the limit of (2.16) where first $r \to \infty$ and then $M \to \infty$, obtaining

$$\lim_{r\to\infty} (\psi - \psi^{\text{AB}}) r^{1/2} \exp(-\mathrm{i}kr) = f(\theta-\theta_k) \tag{2.19}$$

where $f = \lim_{M\to\infty} f_{M-1}$. On using the well known asymptotic form of the AB solution, valid when $kr[1-\cos(\theta-\theta_k)] \gg 1$

$$\psi^{\text{AB}} \sim \exp[-\mathrm{i}\alpha(\theta-\theta_k+\pi)] \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) + \exp(\mathrm{i}kr) r^{-1/2} f^{\text{AB}}(\theta-\theta_k) \tag{2.20}$$

we find from (2.19)

$$\psi \sim \exp[-\mathrm{i}\alpha(\theta-\theta_k+\pi)] \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) + \exp(\mathrm{i}kr) r^{-1/2} F_k^\alpha(\theta-\theta_k) \tag{2.21}$$

where the total scattering amplitude is given by

$$F_k^\alpha(\theta-\theta_k) = f^{\text{AB}}(\theta-\theta_k) + \frac{\exp(-\mathrm{i}\pi/4)}{(2\pi k)^{1/2}} \times \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}\pi(n-|n+\alpha|)][\exp(2\mathrm{i}\Delta_n)-1] \exp[\mathrm{i}n(\theta-\theta_k)]. \tag{2.22}$$

In (2.20) and (2.21) the angle $(\theta-\theta_k+\pi)$ must be chosen in the interval $(-\pi, \pi)$. We note that the asymptotic form (2.21) clearly satisfies the conditions which were discussed following (2.7), thus verifying the correctness of the choice (2.8). From our derivation, it is clear that (2.22) also applies for any cylindrically symmetric potential which vanishes for all radii greater than some $a > r_0$, the only difference being that in the functions (2.6), which determine the phase shifts, the argument of all Bessel functions becomes ka and the logarithmic derivative kA_n of the interior solution is evaluated at $r = a$.

For the scattering amplitude f^{AB} applying to the unscreened magnetic string it is sufficient to consider $-\frac{1}{2}\leqslant \alpha \leqslant \frac{1}{2}$ since it follows from (2.11) that

$$\psi^{\mathrm{AB}}(\alpha+N)=\exp[-\mathrm{i}N(\theta-\theta_k+\pi)]\psi^{\mathrm{AB}}(\alpha) \qquad N \text{ any integer.} \tag{2.23}$$

Then the procedure of AB leads to

$$f^{\mathrm{AB}}(\theta-\theta_k)=\frac{\exp(\mathrm{i}\pi/4)}{(2\pi k)^{1/2}}\sin\pi|\alpha|\left(\frac{\exp(\mp\frac{1}{2}\mathrm{i}\theta')}{\cos\frac{1}{2}\theta'}\right)_{\theta'=\theta-\theta_k+\pi} \qquad \text{as } \alpha\gtrless 0 \tag{2.24}$$

where θ' must be chosen in the range $-\pi<\theta'<\pi$. In fact, the derivation of (2.24) is valid in the greater range $-1\leqslant\alpha\leqslant 1$, as also follows from (2.23). However it differs by a constant phase factor from the scattering amplitude of AB (equation (21)) (but agrees with that of Kawamura *et al* (1982, footnote to p 1273)). The discrepancy† can be resolved by observing that, for the $\mathrm{e}^{-\mathrm{i}r'}$ terms to vanish from the AB equation (21), the $(-\mathrm{i})^{1/2}$ factor in their equation (20) must be interpreted as $\exp(\mathrm{i}3\pi/4)$. When this is done the resulting scattering amplitude becomes identical to (2.24).

From (2.22) and (2.24) we may evaluate the momentum transfer cross section (per unit length of solenoid)

$$\sigma_{\mathrm{tr}}^{\alpha}(k)=\int_0^{2\pi}(1-\cos\phi)|F^{\alpha}(\phi)|^2\,\mathrm{d}\phi. \tag{2.25}$$

The physical interpretation of this quantity is such that the force component in the incident direction $\hat{k}$, which is exerted on unit length of the solenoid by an incident beam of unit particle flux, is $mv\sigma_{\mathrm{tr}}$ where $v=\hbar k/m$ is the speed of the incident and radially scattered particles at large distances from the solenoid. That this familiar interpretation also holds in the present case, where the vector potential enters the velocity operator, may be shown by considering the scattering of wavepackets, for example, by a procedure similar to that of Messiah (1961, ch X, §§ 4, 5 and 6, attributed to Chew and Low).

The calculation of (2.25) is straightforward, if tedious, and yields

$$\sigma_{\mathrm{tr}}^{\alpha}(k)=(2/k)(S^{\alpha}+P^{\alpha})=\sigma_{\mathrm{tr}}^{-\alpha}(k) \tag{2.26}$$

where

$$S^{\alpha}=S^{-\alpha}=\sum_{n=-\infty}^{\infty}\sin^2(\Delta_n-\Delta_{n-1}) \tag{2.27}$$

and

$$P^{\alpha}=P^{-\alpha}=\sin^2\pi\alpha+2\sin\pi|\alpha|\sin(\pi|\alpha|+\Delta_{\mp1}-\Delta_0)$$
$$\times[2\sin\Delta_0\sin\Delta_{\mp1}+\sin(\Delta_0+\Delta_{\mp1})]. \tag{2.28}$$

The expression for P^{α} is valid if $-1\leqslant\alpha\leqslant 1$ and the upper (lower) signs are taken according to whether $\alpha>0(<0)$. Note that in general P^{α} is not periodic in α and expressions valid in other ranges of α follow from (2.28) on replacing Δ_0 and $\Delta_{\mp1}$ by higher-order phase shifts. The fact that (2.26)-(2.28) are even functions of α (all ranges) follows from the symmetry (2.10*b*).

† The question of correctness of (2.24), both with respect to amplitude and phase, is most important in view of the interference between the two terms of (2.22) and the effects of such interference on, e.g., the momentum transfer cross section.

We proceed to examine the implications of (2.26)-(2.28) for the 'enclosed-flux' case, $V\to\infty$.

3. Enclosed flux ($V\to\infty$): momentum transfer

Our aim is to show that even when the flux is surrounded by an impenetrable barrier it still affects the scattering and, more particularly, the momentum transfer. For $V\to\infty$, the phase shifts are given by (2.17) and (2.6) as (cf Kretzschmar 1965)

$$\tan \Delta_n^\alpha = J_{|n+\alpha|}(kr_0)/Y_{|n+\alpha|}(kr_0). \tag{3.1}$$

It is evident from (2.22) that the scattering amplitude retains a flux dependence even in this extreme case for which the wavefunction vanishes at all points $r<r_0$ where the flux density is non-zero. For example, when $kr_0 \ll 1$ and $|\alpha|\ll 1$, (3.1) yields

$$\tan \Delta_{n\neq 0}^\alpha = \frac{-\pi|n+\alpha|}{[\Gamma(1+|n+\alpha|)]^2}\left(\frac{kr_0}{2}\right)^{2|n+\alpha|} \tag{3.2a}$$

and therefore negligible (if we assume $k^2r_0^2 \ll |\alpha|$) while

$$\tan \Delta_0^\alpha = \sin \pi|\alpha|\left[\cos \pi|\alpha| - \left(\frac{2}{kr_0}\right)^{2|\alpha|}\frac{\Gamma(1+|\alpha|)}{\Gamma(1-|\alpha|)}\right]^{-1}. \tag{3.2b}$$

Expanding the latter expression for small $|\alpha|$ yields

$$\tan \Delta_0^\alpha = \tan \Delta_0^0\left(1+\frac{\pi|\alpha|}{\sin 2\Delta_0^0}+\mathrm{O}(\alpha^2)\right) \tag{3.3}$$

where

$$\Delta_0^0 = \tan^{-1}\frac{\pi/2}{\gamma+\ln(kr_0/2)} \tag{3.4}$$

is the phase shift appropriate to the flux-free cylinder. These expressions show how the AB solution ($\Delta_n^0\equiv 0$) evolves in the limit $kr_0\to 0$ when $\Delta_0^0\to 0$. They can also be used to study the scattering amplitude and the momentum transfer cross section in the long-wavelength limit. We go on to consider the practically more important short-wavelength region, $kr_0\gg 1$.

We observe that (3.1) implies

$$\Delta_n^{\alpha+1} = \Delta_{n+1}^\alpha \tag{3.5}$$

and then (2.12) yields (cf (2.23); see also Berry *et al* (1980))

$$\psi(\alpha+N) = \exp[-\mathrm{i}N(\theta-\theta_k+\pi)]\psi(\alpha) \qquad N=\text{integer}. \tag{3.6}$$

It follows that $\sigma_{\mathrm{tr}}^{\alpha+N}=\sigma_{\mathrm{tr}}^\alpha$ so it is sufficient to consider henceforth $0\leqslant\alpha\leqslant 1$. Adopting (3.1) we find after some manipulations involving use of (Watson 1944, § 3.63)

$$J_\nu(z)\,Y_{\nu+1}(z) - J_{\nu+1}(z)\,Y_\nu(z) = -2/\pi z \tag{3.7}$$

that (2.27) may be written, when $kr_0\gg 1$,

$$S^\alpha - S^0 = (2/\pi kr_0)^2(T^\alpha - T^0) - \sin^2 \pi\alpha \tag{3.8}$$

where

$$T^\alpha = \sum_2^\infty \left(\frac{1}{Q_{m+\alpha}Q_{m-1+\alpha}} + \frac{1}{Q_{m-\alpha}Q_{m-1-\alpha}} \right) \tag{3.9}$$

with

$$Q_\nu = Q_\nu(kr_0) \equiv J_\nu^2(kr_0) + Y_\nu^2(kr_0). \tag{3.10}$$

In obtaining (3.8) we have also used

$$\Delta_n = \frac{3\pi}{4} + \frac{\pi}{2}|n+\alpha| - kr_0\left(1 + \frac{4|n+\alpha|^2 - 1}{8(kr_0)^2} + \ldots\right) \qquad kr_0 \gg n^2 \tag{3.11}$$

which follows† from (3.1).

On using (3.11) for Δ_0 and Δ_{-1}, (2.28) becomes

$$P^\alpha(0 \leq \alpha \leq 1) = 3\sin^2 \pi\alpha - 2\sqrt{2}\sin \pi\alpha \cos(2kr_0 - \pi/4). \tag{3.12}$$

To evaluate S^α one could follow Morse and Feshbach (1953, p 1380) and adopt the asymptotic form (3.11) for those orders $|n| \leq kr_0$ which are (see § 4) effective in determining $\sigma_{\text{tr}}^\alpha$. This procedure would be acceptable if only $\sigma_{\text{tr}}^\alpha$ were required but for calculating the small difference $(\sigma_{\text{tr}}^\alpha - \sigma_{\text{tr}}^0)$ the neglect of terms $|n| \geq kr_0$ cannot be taken lightly. Adopting it one finds $S^\alpha - S^0 = -\sin^2 \pi\alpha$, a result which seems plausible but which we have not succeeded in proving. However, by proceeding as follows we are able to prove inequalities sufficient for our purposes. We first note that (Watson 1944, § 13.73).

$$Q_\nu(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z \sinh t)\cosh 2\nu t \, dt \tag{3.13}$$

where the modified Bessel function is defined by (Watson 1944, § 13.74)

$$K_0(z) = \int_0^\infty \exp(-z\cosh t)\, dt > 0. \tag{3.14}$$

It follows that $Q_\nu(z)$ is an increasing function of $\nu > 0$ and using this property it is easy to show from (3.9) that

$$-(\pi kr_0/2)^2 < T^\alpha - T^0 < (\pi kr_0/2)^2 \qquad 0 \leq \alpha \leq 1,\ kr_0 \gg 1 \tag{3.15}$$

and hence, using (3.8),

$$-1 - \sin^2 \pi\alpha < S^\alpha - S^0 < 1 - \sin^2 \pi\alpha \qquad 0 \leq \alpha \leq 1,\ kr_0 \gg 1. \tag{3.16}$$

These inequalities are evidently very crude, at least near $\alpha = 0$, 1 where $S^\alpha - S^0 = 0$. However, putting $\alpha = \frac{1}{2}$ and combining (3.16) with (3.12) we find for the momentum

† The easiest way of obtaining (3.11) is to observe, using (3.1), that $\Delta_n = \pi/2 - \theta_{|n+\alpha|}$ where θ_ν is defined by Abramowitz and Stegun (1965, equation (9.2.17)). The asymptotic expansion (3.11) is then taken from their equation (9.2.29).

transfer cross section (2.26) when $kr_0 \gg 1$

$$\frac{2}{k}[3-2\sqrt{2}\cos(2kr_0-\pi/4)] > \sigma_{\rm tr}^{1/2}(k)-\sigma_{\rm tr}^{0}(k)$$
$$> \frac{2}{k}[1-2\sqrt{2}\cos(2kr_0-\pi/4)]. \tag{3.17}$$

With r_0 of the order of tens of microns and electron wavelengths of the order of 10^{-10} m we have $kr_0 \sim 10^4$†. Averaging over a small spread in either electron wavelength or geometrical dimension will then give rise to an observed cross section $\sigma_{\rm tr}^{1/2}$ related to that observed in the absence of flux by

$$6/k > \sigma_{\rm tr}^{1/2}(k)-\sigma_{\rm tr}^{0}(k) > 2/k \tag{3.18}$$

i.e. there is an additional‡ periodic flux-dependent force, per unit length of solenoid and parallel to the incident beam, whose peak-to-peak amplitude, $f_{\rm pp}$, satisfies

$$f_{\rm pp} > mvJ_{\rm p}2/k = 2\hbar J_{\rm p} \qquad kr_0 \gg 1 \tag{3.19}$$

where $J_{\rm p}$ is the particle flux in the incident beam. The appearance of $\hbar$ in (3.19) indicates the quantum nature of this effect. It is remarkable that (3.19) does not depend on either wavelength or solenoid radius (in the domain $kr_0 \gg 1$).

If we leave the solenoid radius fixed at any radius r_0 and place the impenetrable shield at radius $a > r_0$ the above argument is unaltered except that r_0 is replaced by a. The implications of (3.19) will be discussed in § 5.

4. Finite screening barriers

Our deduction of (3.19) rests on the use of (3.1) for the phase shifts and we need to examine the extent to which our conclusions depend on this adoption of the impenetrable-barrier model, $V\to\infty$, which is 'unphysical' to the extent that it represents a limit which may be approached but not reached. If we take it as granted that $\sigma_{\rm tr}^{\alpha}$ for the real physical situation (finite V) does actually possess a limit as $V\to\infty$ it is hard to imagine how it could differ from the cross section calculated with phase shifts (3.1). Nevertheless in the case of finite V, the adoption of (3.1) for *all n* is questionable and some consideration of its consequences is in order. The questions extend to the *periodicity* of the cross section also, since this was deduced from the symmetry (3.5) which is applicable (only?) to the AB case $\Delta_n \equiv 0$ and to the impenetrable-barrier model. We first show that only those phase shifts Δ_n up to order $|n| \approx kr_0 \gg 1$ need be considered§. Consider $\alpha = 0$: from (3.8) we see that it will suffice to show that

$$t_\varepsilon(kr_0) \equiv \left(\frac{2}{\pi kr_0}\right)^2 \sum_{(1+\varepsilon)kr_0}^{\infty} \frac{1}{Q_m Q_{m-1}} \tag{4.1}$$

† For the interference experiments of Bayh (1962), $kr_0 \sim 10^7$.

‡ In the final paragraph of § II F of their review article OP refer to such forces as 'secondary' effects. This refers to the fact that the 'primary' terms in their wavefunction (2.137) reproduce the cross section σ^0 of the zero-flux case; the terms responsible for (3.19) are neglected.

§ The same conclusion is reached by Morse and Feshbach (1953, p 1380) in the case $\alpha = 0$, but the arguments used by these authors are not sufficiently precise for our purposes.

is small compared to unity when $\varepsilon > 0$ is small compared to unity. To prove this observe that when $\nu > x > 0$, $Y_\nu(x) < 0$ and $Y'_\nu(x) > 0$; both these results follow from the asymptotic form (A1.3) and the fact that there are no zeros of the functions $Y_\nu(x)$ or $Y'_\nu(x)$ for $x \leqslant \nu$ (see Watson 1944, § 15.81). It then follows from the recurrence relations for the Y (Watson 1944, § 3.56) that

$$|Y_{\nu+1}(x)| > \frac{\nu+1}{x}|Y_\nu(x)| \qquad \nu > x > 0. \tag{4.2}$$

Applying this relation successively one shows

$$0 < t_\varepsilon(kr_0) < \frac{(2/\pi kr_0)^2}{|Y_{kr_0}(kr_0)|^4} \frac{(1+\varepsilon)^2}{(1+\varepsilon)^4 - 1}[1 + \mathrm{O}(1/kr_0)]. \tag{4.3}$$

On using (Watson 1944, § 8.2)

$$Y_\nu(\nu) \sim -3^{1/3}\Gamma(1/3)/2^{2/3}\pi\nu^{1/3} \tag{4.4}$$

inequality (4.3) becomes

$$0 < t_\varepsilon(kr_0) < \left(\frac{2^{7/6}\pi^{1/2}}{3^{1/3}\Gamma(\frac{1}{3})}\right)^4 \frac{1}{(kr_0)^{2/3}} \frac{(1+\varepsilon)^2}{(1+\varepsilon)^4 - 1}(1 + \mathrm{O}(1/kr_0)). \tag{4.5}$$

This is small compared to unity if $(kr_0)^{-2/3} < \varepsilon \ll 1$, showing that phase shifts of order greater than $[kr_0 + \mathrm{O}((kr_0)^{1/3})]$ may be disregarded in calculating σ^0_{tr}. The same argument applies when $\alpha \neq 0$, showing that only those phase shifts such that $|n+\alpha| \leqslant kr_0$ need be considered, e.g. it is sufficient to retain only $|n+\alpha| \leqslant 2kr_0$. Now for all finite n it follows from (2.6) that as $V \to \infty$, Δ_n approximates the limit (3.1) according to†

$$\tan \Delta_n = (1 + \eta_n) J_{|n+\alpha|}(kr_0)/Y_{|n+\alpha|}(kr_0) \tag{4.6}$$

where

$$\eta_n = \mathrm{O}\left(\frac{\max(1, |\alpha|)}{kr_0 v^{1/2}}\right) + \mathrm{O}\left(\frac{n\alpha}{k^2 r_0^2 v}\right). \tag{4.7}$$

Differentiating, we find for the error introduced into $\sin^2(\Delta_n - \Delta_{n-1})$ by the term η_n in (4.6)

$$\begin{aligned}\delta[\sin^2(\Delta_n - \Delta_{n-1})] &= (\eta_n - \eta_{n-1}) \sin 2(\Delta_n - \Delta_{n-1}) \\ &\quad + (\eta_n - \eta_{n-1})^2 \cos 2(\Delta_n - \Delta_{n-1}).\end{aligned} \tag{4.8}$$

Since almost all significant contributions come from $(\Delta_n - \Delta_{n-1})$ close to π (Morse and Feshbach 1953, p 1380, see also (3.11)) the error introduced into the sum (2.27) when $v < \infty$ is, from (4.8) and (4.7), at most

$$\delta S \approx -kr_0(\alpha/kr_0\sqrt{v})^2. \tag{4.9}$$

This leads to an error in σ_{tr} of order $k^{-1}\delta S$ which is small compared to the magnitude of the magnetic increment (3.18) provided

$$v \equiv V/(\hbar^2 k^2/2m) \gg \alpha^2/kr_0. \tag{4.10}$$

† The study of this limit is sketched in the paragraph immediately preceding equation (5.8) of Brown (1985). The inequality attached to the latter equation (which assumes $|\alpha| \ll 1$ and $v \gg 1$) should be $kr_0 v^{1/2} \gg \max(1, |n\alpha|^{1/2})$.

From the above discussion it is clear that the force (3.19) still exists for a finite (repulsive) potential barrier but that the range of α over which the force is periodic is limited to $\alpha \ll \alpha_{\max} \sim (vkr_0)^{1/2}$. This condition does not impose serious constraints, as discussed below.

We conclude this section by briefly considering the possibility of observing such a force. We start by considering the parameters appropriate to the interference experiments of Bayh (1962). Bayh used a coil of radius $r_0 \approx 10\,\mu$m and 40 keV electrons with $\lambda \approx 0.06$ Å, resulting in $kr_0 \approx 10^7$. Relatively high energies are necessary if the wavenumber spread, caused mainly by work-function variations, is to be reduced below the value (here $\Delta k/k \sim 10^{-7}$) at which the interference effects disappear due to phase smearing. In our case (see the discussion following (3.17)) such phase smearing need not be avoided and much smaller wavenumbers can be considered. Energies as low as $E \equiv \hbar^2k^2/2m \approx 10$ eV corresponding to $kr_0 \approx 10^3$–10^4 should be usable. Then a repulsive barrier of order 1 kV is effectively impenetrable and, according to (4.10), the incremental magnetic force should display its characteristic periodicity over $\alpha_{\max} \sim 10^2$–10^3 cycles as the flux is increased, for example, by applying a steadily increasing (sawtooth) current. We emphasise that *this periodicity is the unique signature* of the sought-after quantum force, *distinguishing it from classical effects* due to $\boldsymbol{B}$ or due to the induced $\boldsymbol{E}$ field arising from the time varying $\boldsymbol{B}$ field. The difficulty of directly observing the quantum force becomes evident when we consider the magnitudes of the forces involved. A beam current of 1 mA over an area ~ 1 cm^2 corresponds to a particle flux $J_p \sim 10^{20}$ m^{-2} s^{-1} which produces a steady (apart from statistical fluctuations) force per unit length $F_0 = r_0\hbar k J_p \sim 10^{-9}$ N m^{-1} on which is superposed a periodic contribution whose amplitude, according to (3.19), is $f \sim \hbar J_p \sim 10^{-14}$ N m^{-1}. We note that this latter contribution does not depend on kr_0, but of course the mass-dependent response of the system decreases with increasing r_0.

The steady F_0 can be automatically accommodated through the elasticity of the suspension but the smallness of f makes detection of the quantum force appear unlikely unless one can find a microscopic analogue system which displays the effect. On the other hand, the periodicity in α, which bestows a periodicity in time, the frequency of which is widely adjustable (by choosing the rate of change of current and the coil parameters) and may be matched to that of mechanical resonance, are features which favour the experimenter. We also observe that our analysis was shown to apply not only to the potential (2.3) but to an arbitrary screening potential (of finite range). As a result we can dispense with the screening capacitor envisaged in § 2 and seek the effects using an unscreened ferromagnetic filament or whisker, excited by an externally applied field.

5. Locality—the role of $\boldsymbol{E}$ and $\boldsymbol{B}$

From the discussion following (3.19) we see that there is an additional force per unit length acting on the screening barrier at radius a, due to quantum effects of the flux contained in the region $r < r_0 \leqslant a$, even when the barrier is 'impenetrable'. This force increment is periodic in α (unit period) and its amplitude is indicated in (3.19). This confirms the remarkable prediction of AB of force effects due to enclosed flux. Since it persists undiminished as the screening radius, a, increases beyond the solenoid radius, r_0, the effect is *manifestly not due to any penetration of the flux-containing region by the wavefunction.* Although this is a vivid example of the 'indirect' action of enclosed flux on charges which are excluded from it, *the force has a strictly local origin*; it is an *electrostatic* force, due to the $\boldsymbol{E}$ field associated with the screening barrier and is seated in this barrier—as we will show after investigating the role of the electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$ respectively) in general.

Because of the long-range nature of a solenoid's magnetic vector potential the scattering amplitude diverges as the scattering angle $(\theta-\theta_k)$ approaches zero, according to (see (2.22) and (2.24))

$$|F_k^\alpha(\theta-\theta_k)|^2 \underset{\theta\to\theta_k}{\sim} \frac{\sin^2\pi\alpha}{2\pi k}\,\frac{1}{\sin^2\frac{1}{2}(\theta-\theta_k)}. \tag{5.1}$$

As a result the total scattering cross section is infinite and the optical theorem does not apply. However the momentum transfer factor in (2.25) removes this singularity in the small-angle scattering, resulting in a finite momentum transfer cross section. The latter is related to the distribution of electric and magnetic field strengths by

$$\sigma_{\mathrm{tr}}^\alpha(\boldsymbol{k})=\langle\boldsymbol{F}\rangle\cdot\hat{\boldsymbol{k}}/2E_k \tag{5.2}$$

where

$$\langle\boldsymbol{F}\rangle=q\int\psi_k^*[\boldsymbol{E}+\tfrac{1}{2}(\boldsymbol{v}\times\boldsymbol{B}-\boldsymbol{B}\times\boldsymbol{v})]\psi_k\,\mathrm{d}^2r \tag{5.3}$$

is the expectation value of the Lorentz force (Schiff 1955, § 23), with respect to the extended scattering state with asymptotic behaviour (2.21), and $\boldsymbol{v}$ is the velocity operator

$$\boldsymbol{v}=-\mathrm{i}\hbar\nabla-q\boldsymbol{A}. \tag{5.4}$$

The result (5.2), which may be regarded as a generalisation of both the optical theorem and of Ehrenfest's theorem, is established for solenoid scattering in appendix 2. The proof, which is based on that of Brown (1986), assumes that $\boldsymbol{E}$, $\boldsymbol{B}$, ψ and $\nabla\psi$ are everywhere well defined, so that pseudo-physical concepts such as 'infinite' potential barriers must be avoided or, at least, treated with proper care, e.g. by considering large, but finite, barriers.

The magnetic-string solution due to Aharonov and Bohm (1959, equation (22)) corresponds to a momentum transfer cross section of $2/k$. Applying (5.2) and (5.3) in the absence of a screening barrier (i.e. $\boldsymbol{E}\equiv 0$) one sees that if this non-zero σ_{tr} is to be understood the magnetic field cannot consistently be regarded as confined to a (vanishingly small) region in which ψ vanishes. This question is discussed by OP (§ II E) who reach equivalent conclusions by directly calculating the expectation value of the Lorentz force due to field penetration and showing it to account for the momentum exchanged in the scattering process, thus verifying the earlier suggestion of Peshkin *et al* (1961).

If the solenoid is surrounded by a coaxial screening barrier† at radius $a>r_0$ one may arrange, by making a large enough, that the contribution to (5.3) of the magnetic induction term is arbitrarily small. Then the force exerted on the barrier by the scattered particles, although dependent on the magnetic flux because of the latter's appearance in the Hamiltonian (2.1) which determines ψ, is seen to arise *entirely from the electric field* $\boldsymbol{E}$ due to the barrier‡.

† To eliminate long-range electric fields it is convenient to envisage a barrier in the form of coaxial capacitor plates as described in § 2.

‡ Note that even when $V\to\infty$ there must be *some* penetration of the barrier in order to account for $\sigma_{\mathrm{tr}}\neq 0$. This is physically obvious *if* a local scattering force is to be experienced and, irrespective of locality, its truth is evident from (5.2) and (5.3), whether $\alpha=0$ or not.

6. Summary

By applying partial-wave analysis (§ 2) we extended the AB treatment of solenoid scattering to include the effects of screening and non-zero radius. The asymptotics of the wavefunction expressed in (2.21) and (2.22) display formal similarities to formulae appropriate to modified Coulomb scattering.

As the screening potential is made infinite (§ 3) the scattering amplitude and the force exerted by the scattered beam on the solenoid were found to retain a magnetic-flux dependence which becomes periodic in the enclosed flux. The magnitude of the incremental force due to the enclosed flux was estimated (equation (3.19)). In § 4 the force was shown to persist when the condition of idealised impenetrability of the barriers was relaxed and the problems of directly observing it were considered. Finally (§ 5) the role of the local Maxwellian fields was considered and the effects of 'unpenetrated' flux were shown to be manifestations of the non-locality imparted to quantum phenomena by the extended nature of the wavefunction.

Regardless of the practicability of observing them we have established a firm theoretical base for the existence of quantum forces due to enclosed fluxes under realistic conditions.

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Appendix 1. The high-order phase shifts and Bessel functions

We study the n dependence of Δ_n^α as $|n|\to\infty$. According to (2.10) and (2.6) Δ_n (we henceforth drop the superscript) depends on A_n of (2.7a). To study the latter we find, directly from the defining series (2.7c),

$$\begin{aligned}\left(\frac{\partial}{\partial z}\ln\Phi\right)_{z=\alpha} &\underset{n\to-\infty}{\sim} 0+\frac{(\frac{1}{2}-\zeta/\alpha)}{1+|n|}(1+\alpha|n|^{-1}+\mathrm{O}(n^{-2}))\\ &\underset{n\to+\infty}{\sim} 1-\frac{(\frac{1}{2}+\zeta/\alpha)}{1+|n|}(1-\alpha|n|^{-1}+\mathrm{O}(n^{-2}))\end{aligned} \tag{A1.1}$$

where we have written $\zeta=k^2r_0^2(1-v)/4$. On using (A1.1) in (2.7a) we find, subject to the restriction $|n|\gg\max\{1,|\alpha|,|\zeta|\}$,

$$kr_0A_n \underset{|n|\to\infty}{\sim} |n+\alpha|-n^{-1}(\alpha+2\zeta|n|n^{-1})(1+|n|n^{-2}-\alpha n^{-1})+\mathrm{O}(n^{-3}). \tag{A1.2}$$

Further, using equations (3) and (5) of Watson (1944, § 8.4), we have

$$
\begin{aligned}
J_\nu(kr_0) &\underset{\nu\to\infty}{\sim} (2\pi\nu)^{-1/2}(ekr_0/2\nu)^\nu(1+\mathrm{O}(\nu^{-1})) \\
Y_\nu(kr_0) &\underset{\nu\to\infty}{\sim} (2\pi\nu)^{+1/2}(ekr_0/2\nu)^{-\nu}(1+\mathrm{O}(\nu^{-1})).
\end{aligned}
\tag{A1.3}
$$

The leading terms of (A1.3) furnish good approximations for $\nu \gg \max\{1, k^2r_0^2\}$ and corresponding expressions for J' and Y' follow by differentiating or using the appropriate recurrence relations. The above expressions when used in (2.6) and (2.10) show that $\Delta_n \to 0$ as $|n| \to \infty$ according to

$$
|\Delta_n| < |\tan \Delta_n| \sim \frac{\pi(kr_0/2)^{2|n+\alpha|}}{2|n+\alpha|[\Gamma(1+|n+\alpha|)]^2}|(\alpha+2\zeta|n|n^{-1})|(1+\mathrm{O}(n^{-1})). \tag{A1.4}
$$

The remainder terms in (A1.4) depend on kr_0, α and ζ and the expansion is valid provided $n \gg \mathcal{M} = \max\{1, |\alpha|, |\zeta|, k^2r_0^2\}$. Very crudely (A1.4) implies the existence of $\mathcal{N} \gg \mathcal{M}$ such that

$$
|\Delta_n| < \frac{\pi(kr_0/2)^{2|n+\alpha|}}{|n+\alpha|[\Gamma(1+|n+\alpha|)]^2}|(\alpha+2\zeta|n|n^{-1})| \qquad n > \mathcal{N}. \tag{A1.5}
$$

In order to study the expansion (2.12) we also need to establish suitable bounds on $H^{(1)}_{|n+\alpha|}(kr)$. Starting from the recurrence relation (Watson 1944, § 3.6)

$$
H^{(1)}_{\nu+1}(z) = -H^{(1)}_{\nu-1}(z) + \frac{2\nu}{z}H^{(1)}_\nu(z) \tag{A1.6}
$$

we can prove, for $|z| \geq 2$, real $n \geq 1$ and real $p > 2$

$$
|H^{(1)}_{p+n}(z)| < \frac{(p+n)!}{p!}(|H^{(1)}_p(z)| + |H^{(1)}_{p-1}(z)|). \tag{A1.7}
$$

To establish (A1.7), use (A1.6) with $\nu = p$ and $p+1$ successively to prove the inequality for $n=1$ and 2 respectively. Then use (A1.6) again to prove (A1.7) by induction, assuming it to hold for $H^{(1)}_{p+n-1}$ and $H^{(1)}_{p+n-2}$. Since all Bessel functions satisfy the same recurrence relations the inequality also holds for J, Y and $H^{(2)}$. Now if $\alpha > 0$ we can apply (A1.7) with $p = 2+\alpha > 2$ to obtain

$$
|H^{(1)}_{n+\alpha}(kr)| < \frac{(n+\alpha+2)!}{(\alpha+2)!}(|H^{(1)}_{\alpha+2}(kr)| + |H^{(1)}_{\alpha+1}(kr)|) \qquad kr > 2, n \geq 1, \alpha > 0. \tag{A1.8}
$$

If $\alpha < 0$, put $\alpha = [\alpha] + f_\alpha$ where $0 \leq f_\alpha < 1$ and integer $[\alpha] \leq -1$. Also suppose $n + \alpha \geq 0$ so $n \geq 1$. Then we may apply (A1.7) with $p = 2 + f_\alpha > 2$ to obtain

$$
|H^{(1)}_{2+f_\alpha+n}(kr)| < \frac{(2+f_\alpha+n)!}{(2+f_\alpha)!}(|H^{(1)}_{2+f_\alpha}(kr)| + |H^{(1)}_{1+f_\alpha}(kr)|) \qquad kr > 2, n \geq -[\alpha], \alpha < 0. \tag{A1.9}
$$

The importance of (A1.8) and (A1.9) is that they allow complete separation of the r and n dependence; for using the asymptotic form (2.13) they yield for $kr \gg 1$, $n > |\alpha|$

$$
\begin{aligned}
|H^{(1)}_{n+\alpha}(kr)| &< 2\left(\frac{2}{\pi kr}\right)^{1/2}\frac{(n+\alpha+2)!}{(\alpha+2)!}(1+P_\alpha(r)) \qquad \alpha > 0 \\
|H^{(1)}_{n+\alpha}(kr)| &< 2\left(\frac{2}{kr}\right)^{1/2}\frac{(2+f_\alpha+n)!}{(2+f_\alpha)!}(1+Q_\alpha(r)) \qquad \alpha < 0.
\end{aligned}
\tag{A1.10}
$$

The remainders P_α and Q_α are $\mathrm{O}(r^{-1})$ and do not depend on n. They may be made small by taking kr sufficiently large but it suffices to ensure $|P_\alpha(r)| < 1$, $|Q_\alpha(r)| < 1$, say.

This is assured if $kr > x_\alpha$ where x_α is determined by the behaviour of the finite-order Bessel functions in (A1.8) and (A1.9) but need not be explicitly evaluated. Then (A1.10) become

$$|H^{(1)}_{n+\alpha}(kr)| < 4\left(\frac{2}{\pi kr}\right)^{1/2}\frac{(n+\alpha+2)!}{(\alpha+2)!} \qquad \alpha>0,\ kr>x_\alpha$$
$$|H^{(1)}_{n+\alpha}(kr)| < 4\left(\frac{2}{kr}\right)^{1/2}\frac{(2+f_\alpha+n)!}{(2+f_\alpha)!} \qquad \alpha<0,\ kr>x_\alpha. \tag{A1.11}$$

For the last two terms of (2.15) we find

$$\sum_M^{\pm} = \mathrm{i}\sum_{\pm M}^{\pm\infty} \exp[\mathrm{i}n(\theta-\theta_k+\pi/2)]\mathrm{i}\exp(\mathrm{i}\delta_n)\sin\Delta_n H^{(1)}_{|n+\alpha|}(kr) \tag{A1.12}$$

and it is now trivial to show, using (A1.11) and (A1.5), that $r^{1/2}(\Sigma_M^+ + \Sigma_M^-)$ appearing in (2.16) can be made arbitrarily small by choosing M large enough, this value of M being independent of r.

Appendix 2

The result represented by (5.2) and (5.3) is proved for a broad class of three-dimensional systems in Brown (1986); equations in that paper will be referred to by the prefix I. The extension to two dimensions is trivial but the problems raised by the divergence of the solenoid scattering amplitude at small scattering angles are more substantial. In particular the optical theorem of (I, equation (15)) becomes meaningless.

Consider the problem in three dimensions with the z axis being the axis of the solenoid. It suffices to consider normal incidence ($\boldsymbol{k}\perp\hat{\boldsymbol{z}}$), in which case the wavefunction (2.12) is appropriate. When $\theta\neq\theta_k$ the corresponding asymptotic form (2.21) may be written

$$\psi \underset{r\to\infty}{\sim} \mathrm{e}^{\mathrm{i}G}+\Phi \tag{A2.1}$$

where

$$G = kr\cos(\theta-\theta_k) - \alpha(\theta-\theta_k+\pi) \tag{A2.2}$$

and

$$\Phi = r^{-1/2}\exp(\mathrm{i}kr)t(\boldsymbol{k}, k\hat{\boldsymbol{r}}). \tag{A2.3}$$

We have rewritten the scattering amplitude from state $\boldsymbol{k}$ to $\boldsymbol{k}'$ as $t(\boldsymbol{k},\boldsymbol{k}')$ to conform to the notation of Brown (1986). On introducing the Lorentz force operator $\boldsymbol{F}$ of (I, equation (3)), integrating throughout the volume τ of a circular cylinder of large length L and radius R and using Gauss' theorem, we find (cf I, equation (5)) for each cartesian component $j=1,2$

$$\frac{2m}{L}\langle F_j\rangle_\tau \equiv \frac{2m}{L}\int_\tau \psi^*F\psi\,\mathrm{d}^3r = \left(\int_0^{2\pi}\boldsymbol{Q}_j\cdot\boldsymbol{r}\,\mathrm{d}\theta\right)_{r=R} \tag{A2.4}$$

where (cf I, equation (6))

$$\boldsymbol{Q}_j \equiv \hbar^2[\psi_{,j}\nabla\psi^* - \psi^*\nabla\psi_{,j} + \psi^*\nabla(\psi a_j) - a_j\psi\nabla\psi^* - 2\psi^*\psi a_j\boldsymbol{a} + 2\psi^*\psi_{,j}\boldsymbol{a}]. \tag{A2.5}$$

We have written $\boldsymbol{a}\equiv \mathrm{i}q\boldsymbol{A}/\hbar$ and the commas denote differentiation.

To simplify the right-hand side of (A2.5) we write

$$\int_0^{2\pi} \boldsymbol{Q}_j \cdot \boldsymbol{r}\, \mathrm{d}\theta = I_j + \delta I_j \tag{A2.6}$$

where

$$I_j = \int_{C-\delta\theta} \boldsymbol{Q}_j \cdot \boldsymbol{r}\, \mathrm{d}\theta \qquad \delta I_j = \int_{\theta_k - \frac{1}{2}\delta\theta}^{\theta_k + \frac{1}{2}\delta\theta} \boldsymbol{Q}_j \cdot \boldsymbol{r}\, \mathrm{d}\theta \tag{A2.7}$$

and the former expression denotes the integral over the complete circle from which the sector of width $\delta\theta$, centred on θ_k, has been removed. Since the singularity at $\theta = \theta_k$ is excluded we may use the asymptotic form (A2.1) in I_j. We find that only the first two terms of (A2.5) contribute as $r \to \infty$, and these correspond to

$$I_j \underset{r\to\infty}{\sim} \hbar^2 \int_{\theta_k+\delta\theta/2}^{\theta_k+2\pi-\delta\theta/2} \bigg[2k_j(\boldsymbol{k}\cdot\hat{\boldsymbol{r}}) + 2k^2 \frac{|t|^2}{r}\frac{x_j}{r} + (k + \boldsymbol{k}\cdot\hat{\boldsymbol{r}}) \times \left(k_j \frac{t^*}{r^{1/2}} \exp(\mathrm{i}G - \mathrm{i}kr) + kx_j \frac{t}{r^{3/2}} \exp(-\mathrm{i}G + \mathrm{i}kr) \right) \bigg] r\, \mathrm{d}\theta. \tag{A2.8}$$

In the last term of (A2.8) we substitute G from (A2.2) and use†

$$\exp[\mathrm{i}kr\cos(\theta-\theta_k)] \underset{r\to\infty}{\sim} (2\pi/kr)^{1/2} \{ \exp[\mathrm{i}(kr - \pi/4)]\delta(\theta-\theta_k) + \exp[-\mathrm{i}(kr-\pi/4)]\delta(\theta - \theta_k + \pi) \}. \tag{A2.9}$$

Since $\theta = \theta_k$ is excluded from (A2.8) the first delta function in (A2.9) does not contribute; neither does the second since $(k + \boldsymbol{k}\cdot\hat{\boldsymbol{r}}) = 0$ when $\theta - \theta_k + \pi = 0$. The first term in (A2.8) is an elementary integral and we find (A2.4) becomes

$$\frac{2m}{L\hbar^2} \boldsymbol{k}\cdot\langle \boldsymbol{F}\rangle_\tau \underset{R\to\infty}{\sim} \frac{1}{\hbar^2} k_j \delta I_j - 4Rk^3 \sin\frac{\delta\theta}{2} + 2k^2 \int_{\theta_k+\delta\theta/2}^{\theta_k+2\pi-\delta\theta/2} |t(\boldsymbol{k}, k\hat{\boldsymbol{r}})|^2 \boldsymbol{k}\cdot\hat{\boldsymbol{r}}\, \mathrm{d}\theta \tag{A2.10}$$

where a sum over $j = 1, 2$ is implied by repeated indices.

It is not possible to let $\delta\theta \to 0$ in (A2.10) since the final term diverges. We remove the divergence by subtracting an integral with the same singularity, as follows. From the continuity of total particle flux‡ (or from the identity (I, equation (4)) in which K is put equal to the identity operator and Gauss' theorem is applied) we have, cf (A2.6)

$$J + \delta J = 0 \tag{A2.11}$$

where

$$J = \int_{C-\delta\theta} \boldsymbol{T}\cdot\boldsymbol{r}\, \mathrm{d}\theta \qquad \delta J = \int_{\theta_k-\delta\theta/2}^{\theta_k+\delta\theta/2} \boldsymbol{T}\cdot\boldsymbol{r}\, \mathrm{d}\theta \tag{A2.12}$$

and

$$\boldsymbol{T} \equiv \psi\nabla\psi^* - \psi^*\nabla\psi + 2\mathrm{i}\psi\psi^*\boldsymbol{a}. \tag{A2.13}$$

On using the asymptotic form (A2.1) we find, by similar steps to those which

† Equation (A2.9), which is appropriate to the range $-\pi < \theta - \theta_k + \pi < \pi$ may be deduced by the method of stationary phase, as may its three-dimensional version (I, equation (12)).

‡ It is this conservation of total particle number in the incident and scattered beams which yields the optical theorem, in cases where $t(\boldsymbol{k}, \boldsymbol{k})$ exists.

led to (A2.8),

$$J=-\mathrm{i}\int_{\theta_k+\delta\theta/2}^{\theta_k+2\pi-\delta\theta/2}\left[2\boldsymbol{k}\cdot\hat{\boldsymbol{r}}+2k\frac{|t|^2}{r}+(k+\boldsymbol{k}\cdot\hat{\boldsymbol{r}})\left(\frac{t^*}{r^{1/2}}\exp(\mathrm{i}G-\mathrm{i}kr)+\mathrm{cc}\right)\right]r\,\mathrm{d}\theta. \tag{A2.14}$$

Using (A2.9) we find, cf (A2.8), that the last term in (A2.14) does not contribute as $r\to\infty$ and (A2.11) becomes

$$\mathrm{i}k^2\delta J \underset{R\to\infty}{\sim} 4k^3R\sin\frac{\delta\theta}{2}-2k^3\int_{\theta_k+\delta\theta/2}^{\theta_k+2\pi-\delta\theta/2}|t(\boldsymbol{k},k\hat{\boldsymbol{r}})|^2\,\mathrm{d}\theta. \tag{A2.15}$$

On adding this to (A2.10) we obtain

$$2\int_{\theta_k+\delta\theta/2}^{\theta_k+2\pi-\delta\theta/2}|t(\boldsymbol{k},k\hat{\boldsymbol{r}})|^2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'-1)\,\mathrm{d}\theta'-\frac{2m}{L\hbar^2k^2}\hat{\boldsymbol{k}}\cdot\langle\boldsymbol{F}\rangle_T \underset{R\to\infty}{\sim}\Delta(R,\delta\theta) \tag{A2.16}$$

in which θ' denotes the polar angle of $\boldsymbol{k}'$ and

$$\Delta(r,\delta\theta)=\int_{\theta_k-\delta\theta/2}^{\theta_k+\delta\theta/2}\Bigg(\frac{\mathrm{i}}{k}(\psi\nabla\psi^*-\psi^*\nabla\psi)$$
$$-\frac{1}{k^3}[(\boldsymbol{k}\cdot\nabla\psi)\nabla\psi^*-\psi^*\nabla(\boldsymbol{k}\cdot\nabla\psi)-(\boldsymbol{k}\cdot\boldsymbol{a})\psi\nabla\psi^*]\Bigg)\cdot\boldsymbol{r}\,\mathrm{d}\theta. \tag{A2.17}$$

The singularity in $t(\boldsymbol{k},\boldsymbol{k}')$ is (see (2.22) and (2.24)) of type $1/\sin\frac{1}{2}(\theta-\theta_k)$ so that the corresponding singularity in $|t|^2$ is cancelled by $(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'-1)\equiv-2\sin^2\frac{1}{2}(\theta-\theta_k)$. Consequently the first term on the left-hand side of (A2.16) has a finite limit as $\delta\theta\to0$ which we now investigate. This limit cannot be evaluated by letting $\delta\theta\to0$ independent of R since (A2.16) depends on the asymptotic form (A2.1), which is valid only when $kR[1-\cos(\theta-\theta_k)]\gg1$, i.e. when

$$\delta\theta\gg R^{-1/2}. \tag{A2.18}$$

We must therefore let $R\to\infty$ and $\delta\theta\to0$ together and to achieve this we may suppose

$$\delta\theta\sim R^{\eta-1/2}\qquad 0<\eta<\tfrac{1}{2}. \tag{A2.19}$$

With $\delta\theta\to0$ we trivially evaluate the integral (A2.17) to obtain

$$\Delta(R,\delta\theta)=R\delta\theta\left[\frac{\mathrm{i}}{k}\left(\psi\frac{\partial\psi^*}{\partial r}-\psi^*\frac{\partial\psi}{\partial r}\right)+\frac{1}{k^2}\left(\psi^*\frac{\partial^2\psi}{\partial r^2}-\frac{\partial\psi}{\partial r}\frac{\partial\psi^*}{\partial r}\right)\right]_{r=R,\theta=\theta_k}\equiv R\delta\theta P_\psi. \tag{A2.20}$$

As the first term on the left-hand side of (A2.16) depends only on $\delta\theta$ and the second depends only on R, and since the respective limits obviously exist, it follows that the limit of $\Delta(R,\delta\theta)$ exists whenever (A2.19) is obeyed and furthermore that this limit is *independent* of η in the range $0<\eta<\frac{1}{2}$. The only possibility, which establishes (5.2) as the limit of (A2.16), is

$$\lim_{R\to\infty,\delta\theta\to0}\Delta(R,\delta\theta)=0. \tag{A2.21}$$

This may be seen by choosing, e.g., $\eta=\frac{3}{8}$ so that (A2.19) yields $R\delta\theta\sim R^{7/8}$. Then in order that Δ remain finite (A2.20) requires $P_\psi\sim R^{-\varepsilon-7/8}$, $\varepsilon>0$. Finally choosing $\eta=\frac{1}{4}$, say, we have $R\delta\theta P_\psi\sim R^{3/4}P_\psi\sim R^{-\varepsilon-1/8}\to0$.

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